Modified Exponential Series Approximation for the Theodorsen Function

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An improved method is developed for the approximation of unsteady aerodynamics in the time domain by a series of decaying exponentials. The new method is different from the previous procedures in that it consistently accounts for the case when the optimum values of the lag parameters in the exponential series are close to one another. This is achieved by introducing a time-weighted exponential series for the repeated pole case. The method uses a nongradient optimizing procedure. Approximations are presented for Theodorsen's lift deficiency function and results are compared with those of a gradient-based method that was published recently.

Introduction

THE representation of the general motion of an aeroelastic structure requires the availability of the unsteady aerodynamic forces in the time domain. An important feature of these forces is the lag associated with the circulatory wake. Theodorsen[1] employed a lift deficiency function in the reduced frequency domain to represent this effect for the oscillatory flow over an airfoil. Jones[2] used a two-term series of decaying exponentials in the time domain to approximate the effect of circulation for the transient aeroelastic motion and solved for the linear coefficients of the series by using the Fourier transform to convert the transfer function into the frequency domain where it must equal Theodorsen's circulation function. This idea was extended by Dowell[3] who used an exponential series in the time domain to represent the unsteady aerodynamic transfer function and then transformed it into a rational function in the Laplace domain. In contrast with this approach is the conventional least-squares method[4] that begins with a suitable approximation in the Laplace domain and then transforms it into the time domain using the inverse Laplace transform. It is to be noted that, although certain effects, such as the aerodynamic damping and the aerodynamic inertia, can be included in the rational function approximation in the Laplace domain, they are essentially left out in the exponential series approximation in the time domain. However, this difference between the two methods is not present when only the circulatory effect is being approximated, such as the lift deficiency function of Theodorsen.

The accuracy of an exponential series approximation depends crucially on the values of the nonlinear lag parameters that occur in the exponentials. Recently, Peterson and Crawley[5] have shown that, when these parameters are optimized to give a minimum squared error between the exact and the approximate values of the transfer function, the accuracy of the approximation is greatly enhanced. However, the objective-function minima obtained by Peterson and Crawley[5] may not be unique because the gradient-based optimizer is unable to escape the local minima. Also, the frequently encountered cases when the optimum values of two or more lag parameters are nearly the same are mistaken to indicate that the same fit accuracy can be achieved by reducing the number of lag states. Moreover, their inclusion of the numerator as well as the denominator coefficients of the transfer function in the Laplace domain as the free parameters of optimization appears to be a less efficient procedure than the one in which the numerator coefficients are determined by a least-squares fit. The latter option is utilized in the present method, which uses a simplex nongradient optimizer to locate the absolute minima of the objective function. The most significant improvement is achieved by using a consistent optimizing procedure, introduced earlier by the authors,[6] which correctly accounts for the case of repeated lag parameters by employing a new approximation that contains time-weighted exponentials. The time-weighting functions are polynomials in time of one degree less than the multiplicity of the corresponding lag parameters. By using this series, the repeated pole case not only becomes meaningful but also allows an improvement in the fit accuracy with the frequency domain data.

Gradient-Based Method as Compared to the Present Method

The present example compares the gradient-based optimization scheme of Peterson and Crawley,[5] who used the exponential time-series approximation of Dowell[3] as the basic transfer function, with the nongradient optimized scheme of the present method employing the same transfer function. For the purpose of comparison, both methods approximate the Theodorsen circulation function.

Transfer Function

Dowell started with the exponential time-series approximation for the aerodynamic indicial response as follows:

$$\phi(t) = a_0 + \sum_{n=1}^{N} a_n e^{-b_n t} \tag{1}$$

Then, this is converted to its impulse response counterpart by the application of the Fourier transform and multiplication by the factor ik:

$$C(k) = a_0 + \sum_{n=1}^{N} \frac{ika_n}{b_n + ik} \tag{2}$$

where $C(k)$ corresponds to Theodorsen's lift deficiency function for an airfoil in an incompressible, oscillatory flow. Notice that Dowell's impulse response approximation, given by Eq. (2), does not include the aerodynamic damping and inertia terms of the conventional least-squares approximation.[4] In the present example, this is not expected to be relevant since only the Theodorsen function is to be fit.

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Method of Peterson and Crawley

The gradient method of Peterson and Crawley[5] has the following features:

1) a_0 is constrained to be the steady-state value of $C(k)$.

2) All a_n and b_n are taken as the free parameters of the optimization.

3) A Newton-Raphson gradient-based optimizer is used to optimize the numerator and denominator constants a_n and b_n.

4) When a case of repeated optimum lag parameters is observed, further optimization is abandoned because it is interpreted to represent a point where no increase in accuracy will occur by increasing the number of lag states.

Table 1 Exponential time-series approximations for Theodorsen's function

Peterson and Crawley[5]	Present method
One lag state	
$a_0 = 1.0000$	$a_0 = 0.9672$
$a_1 = -0.4542$	$a_1 = -0.4299$
$b_1 = 0.1660$	$b_1 = 0.1851$
Error = 0.0215	Error = 0.0072
Two lag states	
$a_0 = 1.0000$	$a_0 = 0.9962$
$a_1 = -0.4027$	$a_1 = -0.1667$
$b_1 = 0.1297$	$b_1 = 0.0553$
$a_2 = -0.1343$	$a_2 = -0.3119$
$b_2 = 1.2660$	$b_2 = 0.2861$
Error = 0.0109	Error = 0.0005608
Three lag states	
$a_0 = 1.0000$	$a_0 = 0.9994$
$a_1 = -0.1524$	$a_1 = -0.1055$
$b_1 = 0.0490$	$b_1 = 0.0371$
$a_2 = -0.2212$	$a_2 = -0.2879$
$b_2 = 0.2385$	$b_2 = 0.1859$
$a_3 = -0.1088$	$a_3 = -0.1003$
$b_3 = 0.3576$	$b_3 = 0.5886$
Error = 0.00102	Error = 0.0002034
Four lag states	
$a_0 = 1.0000$	$a_0 = 0.9996$
$a_1 = -0.1058$	$a_1 = -0.10624$
$b_1 = 0.0367$	$b_1 = 0.0371$
$a_2 = -0.2877$	$a_2 = -0.30304$
$b_2 = 0.1853$	$b_2 = 0.19142$
$a_3 = -0.0009$	$a_3 = 1.8665$
$b_3 = 0.5681$	$b_3 = 1.1106$
$a_4 = -0.1002$	$a_4 = -1.9386$
$b_4 = 0.5914$	$b_4 = 1.0768$
Error = 0.000420	Error = 0.000186
Five lag states	
$a_0 = 1.0000$	$a_0 = 0.9994$
$a_1 = -0.2919$	$a_1 = -0.11667$
$b_1 = 0.1038$	$b_1 = 0.039369$
$a_2 = -0.1167$	$a_2 = 1768.6$
$b_2 = 0.2270$	$b_2 = 0.34398$
$a_3 = -0.1060$	$a_3 = -596.11$
$b_3 = 1.2649$	$b_3 = 0.34817$
$a_4 = -0.0853$	$a_4 = -1172.9$
$b_4 = 1.6161$	$b_4 = 0.34185$
$a_5 = 0.0899$	$a_5 = 0.0382$
$b_5 = 2.4435$	$b_5 = 1.5874$
Error = 0.007320	Error = 0.000174
Six lag states	
$a_0 = 1.0000$	$a_0 = 0.9994$
$a_1 = -0.3354$	$a_1 = -47.868$
$b_1 = 0.1292$	$b_1 = 0.50408$
$a_2 = -0.0491$	$a_2 = -0.11683$
$b_2 = 0.0769$	$b_2 = 0.03941$
$a_3 = 0.0343$	$a_3 = -3.1611$
$b_3 = 0.1682$	$b_3 = 0.30155$
$a_4 = -0.0268$	$a_4 = 46.380$
$b_4 = 0.2452$	$b_4 = 0.50620$
$a_5 = -0.0303$	$a_5 = 0.03185$
$b_5 = 0.2948$	$b_5 = 1.5319$
$a_6 = -0.1230$	$a_6 = 4.2515$
$b_6 = 1.1911$	$b_6 = 0.34731$
Error = 0.00811	Error = 0.000174

Present Method

The exponential time-series transfer function is incorporated in the present method with the following features:

1) a_0 is unconstrained.

2) Only the nonlinear coefficients b_n are taken to be the free parameters for the optimizer. The linear parameters a_0 and a_n are found by a least-squares curve-fitting process.[6]

3) The simplex nongradient optimizer is used.[7]

4) When the optimum values of two or more lag parameters are nearly the same, the following approximation is used:

$$C(k) = a_0 + \sum_{n=1}^{N_1} \frac{ika_n}{ik + b_n} + \sum_{n=N_1+1}^{N_2} \frac{ika_n}{(ik + b_n)^2} + \dots \tag{3}$$

where N_1 is the total number of distinct lag parameters values, $(N_2 - N_1)$ the number of lag parameter values that are repeated two or more times, and so on. The new approximation has exactly the same number of lag states as the conventional approximation for the same fit accuracy and can be employed in a consistent way in a transient flutter analysis.[6]

Objective Function

The objective function is the sum of the squares of the errors between the exact and approximate impulse response functions at all of the reduced frequencies of interest:

$$J = \sum_{m=1}^{M} H(k_m)\bar{H}(k_m)$$

where

$$H(k) = [F(k) + iG(k)] - [F'(k) + iG'(k)]$$

where $\bar{H}(k)$ is the complex conjugate of $H(k)$, $F(k) + iG(k)$ the exact Theodorsen circulation function data at a set of discrete values of the reduced frequency k, $F'(k) + iG'(k)$ the approximate transfer function given by Eq. (2), and M the number of reduced frequencies at which the fit is desired. The 11 reduced frequencies used in this example are $k = 0.00$, 0.025, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, and 1.0.

Results

The results are available in Tables 1 and 2 and also in Fig. 1. From the tables and the figure, it can be concluded that the present method generally produces minimum values of the objective function that are several orders of magnitude less than those predicted by Peterson and Crawley. The latter

Table 2 Effect of initial lag values on the two methods

	Optimum values	
Initial values	Peterson and Crawley[5]	Present method
$b_1 = 0.2000$	$b_1 = 0.1297$	$b_1 = 0.0553$
$b_2 = 1.0000$	$b_2 = 1.2660$	$b_2 = 0.28607$
	Error = 0.01090	Error = 0.0005608
$b_1 = 0.0455$	$b_1 = 0.05187$	$b_1 = 0.0553$
$b_2 = 0.3000$	$b_2 = 0.2819$	$b_2 = 0.28607$
	Error = 0.00119	Error = 0.0005608
$b_1 = 0.2000$	$b_1 = 0.04900$	$b_1 = 0.03705$
$b_2 = 0.5000$	$b_2 = 0.2385$	$b_2 = 0.18594$
$b_3 = 1.0000$	$b_3 = 0.3576$	$b_3 = 0.58861$
	Error = 0.00102	Error = 0.0002034
$b_1 = 0.0594$	$b_1 = 0.03670$	$b_1 = 0.03705$
$b_2 = 0.2540$	$b_2 = 0.1853$	$b_2 = 0.18594$
$b_3 = 0.6520$	$b_3 = 0.5912$	$b_3 = 0.58861$
	Error = 0.00042	Error = 0.0002034

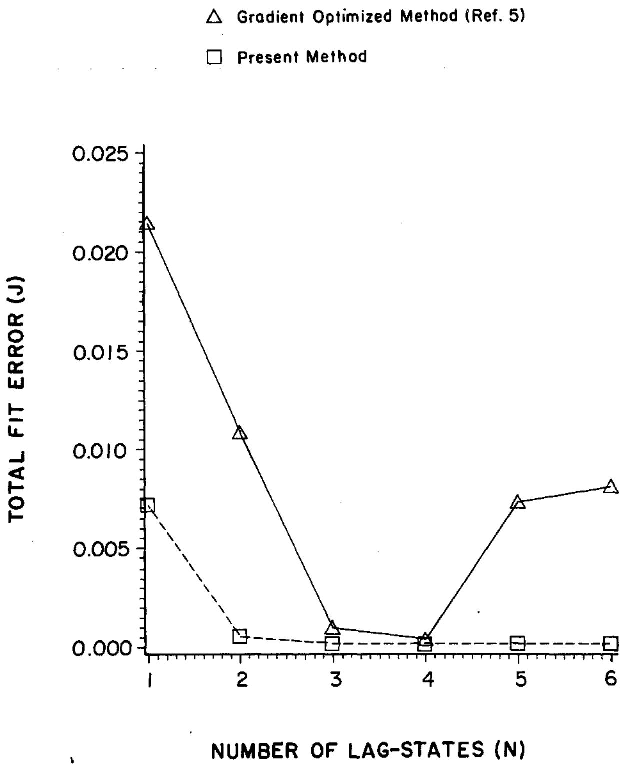

Fig. 1 Variation of the total fit error with the number of lag states (comparison of the gradient method of Ref. 5 with the present method).

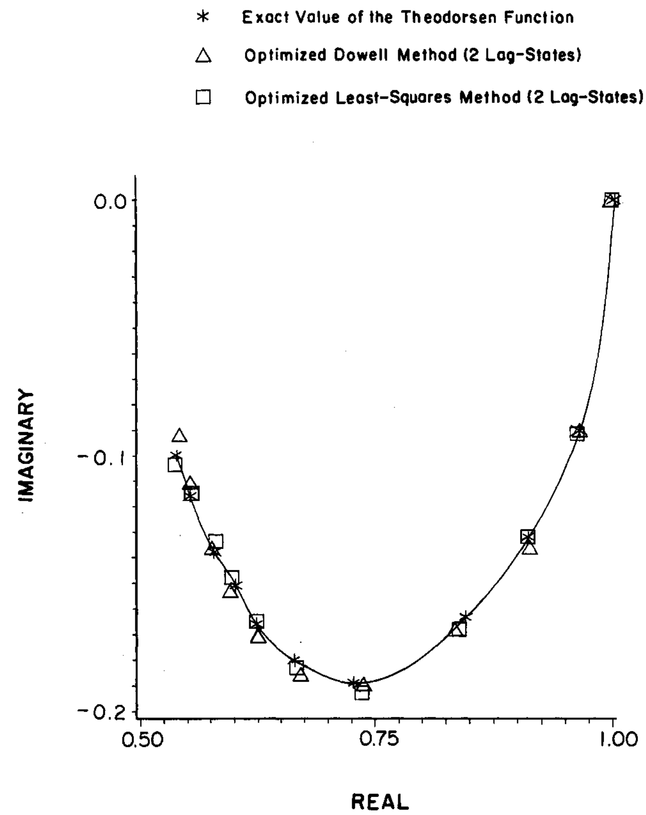

Fig. 2 Exact Theodorsen function as compared with the curve fits for two lag states.

method shows various irregularities, such as giving a greater fit error with an increased number of terms in the transfer function.

Peterson and Crawley constrain the transfer function to match the exact data at zero reduced frequency. This automatically degrades the fit at higher reduced frequencies. The present method imposes no such constraints and is, therefore, able to produce good fits throughout the frequency range of interest. This can be seen by the fact that a_0 comes out to be very close to 1.0, even when it is not constrained to be equal to that value. By including the numerator as well as denominator constants of the transfer function as free parameters for the gradient optimizer, the method of Ref. 5 appears to compromise efficiency because the numerator constants can be solved at one pass by the least-squares curve-fitting process, as in the present method. The effect of starting values of the lag parameters b_n is shown in Table 2. Although the objective function given by the gradient method changes by several orders of magnitude when the starting values of the lag parameters are varied, those of the present method remain invariant with the changing starting conditions. This suggests that the present method is able to locate the absolute minima, and this occurs in all of the cases attempted.

Evident in Table 1 is the phenomenon of repeated poles that occurs for the gradient as well as the nongradient method. Peterson and Crawley[5] did not investigate the significance of the occurrence of two or more lag parameters that have values close to one another. They interpreted this phenomenon by concluding that when two lag parameters are repeated, one could reduce the number of lag states without having much effect on the fit accuracy. This explanation is found to be insufficient by the present investigation, where it is recognized that the simple pole series approximation ceases to be valid in such cases. A multiple pole series is then used to give consistent results while retaining the order of accuracy. If a repeated pole is replaced by only one pole (as suggested in Ref. 5), the

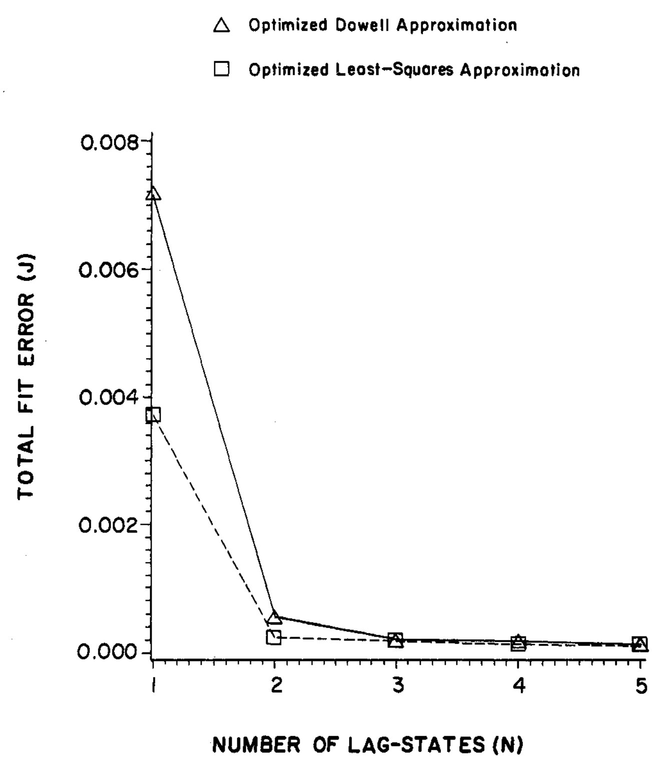

Fig. 3 Variation of the total fit error with the number of lag states (comparison of the optimized Dowell and least-squares methods).

approximation error is found to be several orders of magnitude higher. The new series corresponds to a time-weighted exponential series in the time domain, as opposed to the pure exponential series approximation of Dowell.

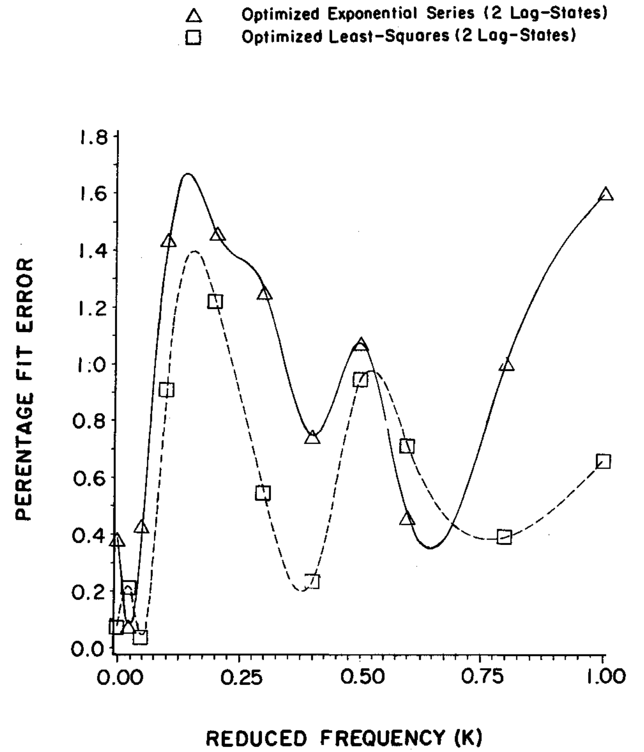

Fig. 4 Variation of the percentage fit error with the reduced frequency (comparison of the optimized exponential series and least-squares methods).

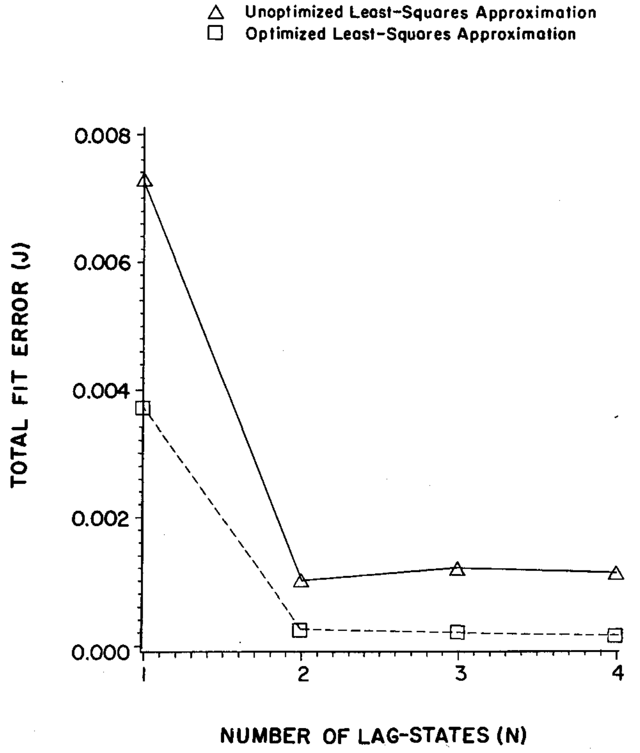

Fig. 5 Variation of the total fit error with the number of lag states (comparison of the optimized and unoptimized least-squares methods).

Dowell's Method and the Least-Squares Approximation

Dowell's transfer function [Eq. (2)] contains only the stiffness and lag terms, whereas the least-squares transfer function has, in addition, the damping and inertia-type terms, as seen in the following expression:

$$C(k) = a_0 + ika_1 + (ik)^2a_2 + \sum_{n=1}^{N} \frac{ika_{(n+2)}}{b_n + ik} \tag{4}$$

Theodorsen's circulation function describes only the lag associated with the wake; therefore, it is reasonable to assume (cf. Jones[2]) that a series of the so-called lag terms would best approximate the Theodorsen function $C(k)$. In other words, having damping and inertia-type terms in the transfer function would not enhance the accuracy of the approximation. Hence, it is expected that, for a given number of lag states, Dowell's approximation would be better than the least-squares approximation. However, the comparisons between the two methods when employed as approximations for the Theodorsen function, seen in Figs. 2–4, indicate otherwise. Figure 2 compares the two fits with the exact data when two lag states are used, and the visual implication is that they are somewhat different, but of nearly equivalent accuracy. Both methods use the same nongradient optimizer. Figure 3 shows that the optimized least-squares fit is better than the optimized Dowell's fit when one (or two) lag parameter is taken. However, as the number of lag states is increased, the difference between the two methods is seen to become small. These comparisons show that, when the number of lag states is small, the optimized least-squares approximation is better than Dowell's for the same number of lag states, even for the case of Theodorsen's function. This is rather surprising since the least-squares method attempts to assign inertia and damping to Theodorsen's function, which does not possess those features. For example, with two lag states,

$$C(k) \simeq 0.54680 - 0.027059(ik) - 0.025222k^2 + \frac{0.005441}{ik + 0.04285} + \frac{0.074729}{ik + 0.22962}$$

The objective function is smaller than that produced by the approximation

$$C(k) \simeq 0.99618 - \frac{0.16666ik}{ik + 0.05530} - \frac{0.31190ik}{ik + 0.28606}$$

The latter is the optimum Dowell's approximation with two lag states and the former is the optimum least-squares approximation with two lag states. The percentage errors of both the approximations are plotted against the reduced frequency in Fig. 4. It should be remembered that the lag state of Dowell's approximation is different from that of the least-squares method used in the present report. The modified least-squares approximation does not contain the Laplace variable in the numerator of the lag state.[6] From the present observations, it must be deduced that a lag state of the form $a_nik/(ik + b_n)$ is not the best approximation of the circulatory lag associated with the Theodorsen function. Figure 5 shows the comparison between optimized and unoptimized least-squares methods. By comparing Figs. 3 and 5, the similarity in the error trends of the unoptimized least-squares method and the optimized Dowell's method can be noticed (except a minor irregularity in the former method when it shows greater fit errors for three and four lag states than for two lag states). However, the unoptimized least-squares fit error is much greater in magnitude than that of the optimized Dowell's method when more than one lag state is used.

Conclusions

The exponential time-series approximation for Theodorsen's function has been optimized using a consistent scheme developed earlier by the authors.[6] The present method is more accurate than a gradient-based approximation published recently. This improvement is due to the correct interpretation of the repeated pole case as one requiring a new approximation for the lag states and also due to the use of a nongradient optimizer. Terms of the form $a_nik/(ik + b_n)$ do not necessarily approximate the circulation function most accurately whereas $a_1ik + a_2(ik)^2 + [a_n/(ik + b_n)]$ is a better approximation when the number of lag parameters b_n is small.

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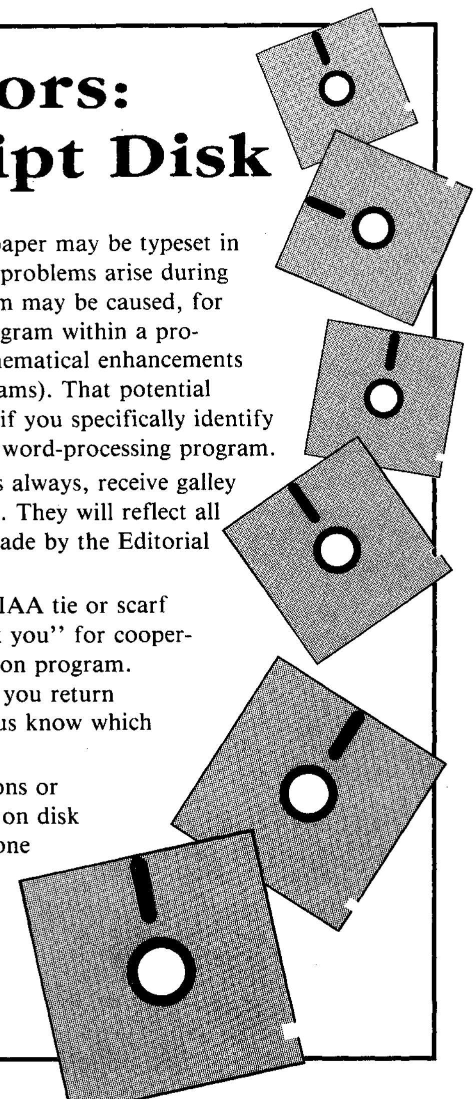

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